

# Algebra of 2D periodic operators with local and perpendicular defects

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## Abstract

We show that 2D periodic operators with local and perpendicular defects form an algebra. We provide an algorithm of finding spectrum for such operators. While the continuous spectral components can be computed by simple algebraic operations on some matrix-valued functions and few number of integrations, the discrete part is much more complicated.

*Keywords:* periodic lattice with defects, Floquet - Bloch spectrum, guided waves, localised waves (states), operator algebras

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## 1. Introduction

Defects in periodic structures play a major role in various fields of science, see, e.g., discussions in [1]. The algebras of multidimensional discrete periodic operators with parallel defects (APD) are considered in the mentioned paper, where the algorithm of finding spectrum based on simple algebraic operations and few number of integration is provided (we call such algorithms as explicit). In the current article we extend two-dimensional APD by adding perpendicular defects. Generally speaking, in comparison with the parallel defects the perpendicular defect makes non-explicit the algorithm of finding point spectrum. At the same time, the continuous components can be computed explicitly in the same way as in [1], [2]. As well as parallel defects, the perpendicular defects have a lot of applications, see, e.g., [3], [4], [5], [6] about electro and optical crossing wave-guides. Some comparison of parallel and perpendicular waveguides is treated in [7]. The methods of finding guided and local waves, and the corresponding spectrum are usually approximative and are based on supercell approaches, where the infinite structure is replaced with a large finite structure which has a discrete spectrum only. In the current paper we propose a non-approximative algorithm of finding spectrum based on an expansion of the periodic operator with defects into the product of the operators with "simple" spectral components.

Let  $M$  be some positive integer. Introduce the following Hilbert space and integral operators

$$L_{2,M}^2 := L^2([0, 1]^2, \mathbb{C}^M), \quad \langle \cdot \rangle_i := \int_0^1 \cdot dk_i, \quad i = 1, 2, \quad \langle \cdot \rangle_{12} := \langle \langle \cdot \rangle_1 \rangle_2, \quad (1)$$

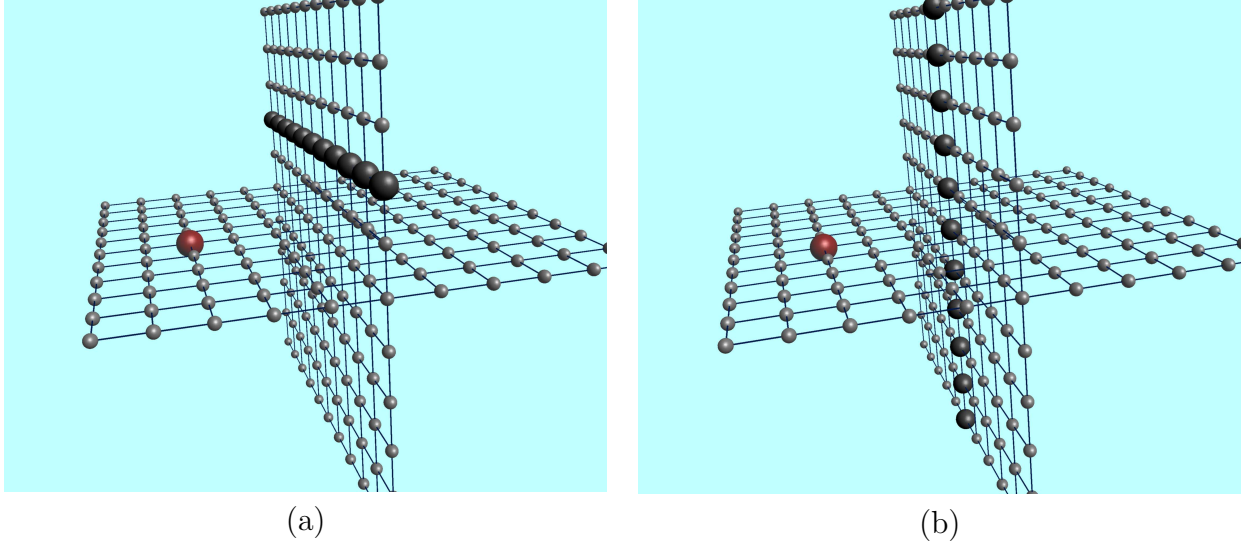


Figure 1: Two coupled 2D lattices with parallel (a) and perpendicular (b) line defects, and one point defect. Periodic operators on the first structure belong to  $\mathcal{H}_{2,M}$ , on the second belong to  $\mathcal{A}_{2,M}$ .

where  $\cdot$  means some matrix- or vector-valued function depending on  $\mathbf{k} = (k_1, k_2) \in [0, 1]^2$ . The bounded and compact operators acting on  $L_{2,M}^2$  will be denoted as  $\mathcal{B}_{2,M}$  and  $\mathcal{K}_{2,M}$  respectively. By analogy with APD (see [1]) define

**Definition 1.1.** *The algebra of 2D periodic operators with local and perpendicular defects*

$$\mathcal{A}_{2,M} = \text{Alg}\{\mathbf{A}\cdot, \langle \cdot \rangle_1, \langle \cdot \rangle_2, \mathcal{K}\} \quad (2)$$

is a minimal non-closed subalgebra of the algebra  $\mathcal{B}_{2,M}$  which contains all operators of multiplication by  $M \times M$  continuous matrix-valued functions  $\mathbf{A}\cdot$ , the integral projectors  $\langle \cdot \rangle_i$ ,  $i = 1, 2$ , and all compact operators  $\mathcal{K} \in \mathcal{K}_{2,M}$ . Here  $\cdot$  denotes the operator argument  $\mathbf{u} \in L_{2,M}^2$ .

Note that APD in 2D case have the form

$$\mathcal{H}_{2,M} = \text{Alg}\{\mathbf{A}\cdot, \langle \cdot \rangle_1, \langle \cdot \rangle_{12}\}, \quad \widetilde{\mathcal{H}}_{2,M} = \text{Alg}\{\mathbf{A}\cdot, \langle \cdot \rangle_2, \langle \cdot \rangle_{12}\}. \quad (3)$$

The general class of multidimensional APD is studied in [1] and [2]. The algebra (2) is an extension of APDs (3), i.e.

$$\mathcal{H}_{2,M} \subset \mathcal{A}_{2,M}, \quad \widetilde{\mathcal{H}}_{2,M} \subset \mathcal{A}_{2,M}. \quad (4)$$

As mentioned above, this extension makes non-explicit the algorithm of finding eigenvalues. The schematic difference between some operators from  $\mathcal{H}_{2,M}$  and  $\mathcal{A}_{2,M}$  is illustrated in Fig. 1. The next theorem is some analogue of the theorem from [1].

**Theorem 1.2.** *Each operator  $\mathcal{A} \in \mathcal{A}_{2,M}$  has a following representation*

$$\mathcal{A}\mathbf{u} = \mathbf{A}_0\mathbf{u} + \mathbf{A}_1\langle\mathbf{B}_1\mathbf{u}\rangle_1 + \mathbf{A}_2\langle\mathbf{B}_2\mathbf{u}\rangle_2 + \mathcal{K}\mathbf{u}, \quad \mathbf{u} \in L_{2,M}^2, \quad (5)$$

where  $\mathcal{K} \in \mathcal{K}_{2,M}$  and  $\mathbf{A}, \mathbf{B}$  are continuous matrix-valued functions on  $[0, 1]^2$  of sizes

$$\dim(\mathbf{A}_0) = M \times M, \quad \dim(\mathbf{B}_j) = M_j \times M, \quad \dim(\mathbf{A}_j) = M \times M_j, \quad j = 1, 2 \quad (6)$$

with some positive integers  $M_j$ . The set of all operators of the form (2) coincides with  $\mathcal{A}_{2,M}$ .

To describe the spectrum of the operators (5) we need the following definition.

**Definition 1.3.** *Let  $\mathcal{A} \in \mathcal{A}_{2,M}$  be some operator of the form (2). Define the following matrix-valued functions (if the inverse matrices exist)*

$$\mathbf{C}_0 = \mathbf{A}_0^{-1}, \quad \mathbf{E}_0 = \mathbf{A}_0, \quad \mathbf{C}_1 = \mathbf{C}_0\mathbf{A}_1, \quad \mathbf{C}_2 = \mathbf{C}_0\mathbf{A}_2, \quad (7)$$

$$\mathbf{E}_1 = \mathbf{I} + \langle\mathbf{B}_1\mathbf{C}_1\rangle_1, \quad \mathbf{E}_2 = \mathbf{I} + \langle\mathbf{B}_2\mathbf{C}_2\rangle_2, \quad (8)$$

$$\mathbf{D}_1(\mathbf{k}, \mathbf{k}') = \mathbf{C}_1(\mathbf{k})\mathbf{E}_1^{-1}(k_2)\mathbf{B}_1(k'_1, k_2)\mathbf{C}_2(k'_1, k_2)\mathbf{B}_2(\mathbf{k}'), \quad (9)$$

$$\mathbf{D}_2(\mathbf{k}, \mathbf{k}') = \mathbf{C}_2(\mathbf{k})\mathbf{E}_2^{-1}(k_1) \int_0^1 \mathbf{B}_2(k_1, k''_2)\mathbf{D}_1(k_1, k''_2, \mathbf{k}')dk''_2, \quad (10)$$

where  $\mathbf{k} = (k_1, k_2)$  and  $\mathbf{k}' = (k'_1, k'_2)$ .

The next Theorem is our main result. It not only provides the explicit procedure of finding inverse operators, but along with the Theorem 1.2 shows that the subset of  $\mathcal{A}_{2,M}$  consisting of all invertible operators  $(\text{Inv}\mathcal{A}_{2,M}, \circ)$  is an algebraic group with the multiplication given by the composition of mappings (the multiplication is the same as in  $\mathcal{A}_{2,M}$ ). Theoretically, it could be that  $\mathcal{A}^{-1} \notin \mathcal{A}_{2,M}$  for  $\mathcal{A} \in \mathcal{A}_{2,M}$ , but it did not happen. Everywhere "inverse" means inverse in the large algebra  $\mathcal{B}_{2,M}$  of all bounded operators.

**Theorem 1.4.** *An operator  $\mathcal{A} \in \mathcal{A}_{2,M}$  of the form (5) is invertible if and only if  $\det \mathbf{E}_j \neq 0$  (7)-(8) everywhere for  $j = 0, 1, 2$  and the operator  $\mathcal{I} + \mathcal{K}_1$  is invertible. The operator  $\mathcal{I}$  is the identity operator and the compact operator  $\mathcal{K}_1$  is defined by*

$$\mathcal{K}_1\mathbf{u} = \int_{[0,1]^2} (\mathbf{D}_2 - \mathbf{D}_1)(\mathbf{k}, \mathbf{k}')\mathbf{u}(\mathbf{k}')d\mathbf{k}' + \mathcal{R} \circ \mathcal{K}\mathbf{u}, \quad \mathbf{u} \in L_{2,M}^2, \quad (11)$$

$$\mathcal{R} = (\mathcal{I} - \mathbf{C}_2\mathbf{E}_2^{-1}\langle\mathbf{B}_2\cdot\rangle_2) \circ (\mathcal{I} - \mathbf{C}_1\mathbf{E}_1^{-1}\langle\mathbf{B}_1\cdot\rangle_1) \circ (\mathbf{C}_0\cdot), \quad (12)$$

where  $\cdot$  means operator argument. Moreover, the inverse operator  $\mathcal{A}^{-1} \in \mathcal{A}_{2,M}$  has the form

$$\mathcal{A}^{-1} = (\mathcal{I} - \mathcal{K}_1 \circ (\mathcal{I} + \mathcal{K}_1)^{-1}) \circ \mathcal{R}. \quad (13)$$

The Theorem 1.4 immediately yields to the next Corollary describing the spectrum.

**Corollary.** *Let  $\mathcal{A} \in \mathcal{A}_{2,M}$  be an operator of the form (5). Taking  $\mathbf{A}_0 := \mathbf{A}_0 - \lambda \mathbf{I}$  ( $\mathbf{I}$  is the identity matrix) in the procedure (7)-(10) we obtain that the spectrum of  $\mathcal{A}$  is*

$$\sigma(\mathcal{A}) = \bigcup_{j=0}^3 \sigma_j \quad \text{with} \quad \sigma_0 = \{\lambda : \det \mathbf{E}_0 = 0 \text{ for some } \mathbf{k} \in [0, 1]^2\}, \quad (14)$$

$$\begin{cases} \sigma_1 = \{\lambda : \det \mathbf{E}_1(k_2) = 0 \text{ for some } k_2 \in [0, 1]\}, \\ \sigma_2 = \{\lambda : \det \mathbf{E}_2(k_1) = 0 \text{ for some } k_1 \in [0, 1]\}, \end{cases} \quad (15)$$

$$\sigma_3 = \{\lambda : \mathcal{I} + \mathcal{K}_1 \text{ is non-invertible}\}. \quad (16)$$

**Remark.** 1) The matrix-valued function  $\mathbf{E}_0$  is defined for any  $\lambda \in \mathbb{C}$ . The matrix-valued functions  $\mathbf{E}_1, \mathbf{E}_2$  are well-defined for  $\lambda \notin \sigma_0$ . More precisely, we can define  $\mathbf{E}_{1,2}$  for some  $\lambda \in \sigma_0$  but it does not affect the spectrum as a set. The analytic (compact-)operator-valued function  $\mathcal{K}_1(\lambda)$  is well-defined for  $\lambda \notin \sigma_0 \cup \sigma_1 \cup \sigma_2$ . So, the procedure of finding spectrum consists of determining  $\sigma_0$ , then  $\sigma_1, \sigma_2$ , and then  $\sigma_3$ .

2) As for the parallel defects (see [1]),  $\sigma_0$  corresponds to non-attenuated eigensolutions,  $\sigma_1, \sigma_2$  correspond to guided eigensolutions, and  $\sigma_3$  are eigenvalues. Note that  $\sigma_0$  does not depend on any perturbation of lower dimension,  $\sigma_1$  and  $\sigma_2$  do not depend on each other and on the compact perturbation  $\mathcal{K}$ .

3) In Corollary, instead of  $\mathbf{A}_0 := \mathbf{A}_0 - \lambda \mathbf{I}$  we may assume a general situation of extended spectral problems where all  $\mathbf{A}_j, \mathbf{B}_j$  somehow depend on the spectral parameter  $\lambda$ .

4) Along with (13) we have the decomposition of the direct operator

$$\mathcal{A} = (\mathbf{A}_0 \cdot) \circ (\mathcal{I} + \mathbf{C}_2 \langle \mathbf{B}_2 \cdot \rangle_2) \circ (\mathcal{I} + \mathbf{C}_1 \langle \mathbf{B}_1 \cdot \rangle_1) \circ (\mathcal{I} + \mathcal{K}_1). \quad (17)$$

Each term belongs to the corresponding subgroup of invertible operators from  $\mathcal{A}_{2,M}$ . This decomposition is unique. The situation is similar to that of [2] except that we probably can not correctly define the vector-valued traces and determinants of  $\mathcal{A}$ .

The work is organized as follows: Section 2 contains proofs of our results. Section 3 provides an application of our results to the problem of wave propagation through 2D spring-mass model with two perpendicular wave-guides. The conclusion is given in Section 4.

## 2. Proof of Theorems 1.2, 1.4

**Proof of Theorem 1.2.** It is obvious that any  $\mathcal{A}$  (5) belongs to  $\mathcal{A}_{2,M}$ . To complete the proof we need to show that the sum and products of the operators of the form (5) have the same form. It is sufficient to show this fact for summands only. For the components from  $\mathcal{H}_{2,M}$  and  $\tilde{\mathcal{H}}_{2,M}$  the corresponding identities are already shown in [1]. It is also obvious that  $\mathcal{K}_{2,M}$  is a two-sided ideal in  $\mathcal{A}_{2,M}$ . It remains to show that:

$$(\mathbf{A}_1 \langle \mathbf{B}_1 \cdot \rangle_1) \circ (\mathbf{A}_2 \langle \mathbf{B}_2 \mathbf{u} \rangle_2) = \int_{[0,1]^2} \mathbf{D}(\mathbf{k}, \mathbf{k}') \mathbf{u}(\mathbf{k}') d\mathbf{k}' \quad (18)$$

is obviously a compact operator with the continuous kernel

$$\mathbf{D}(\mathbf{k}, \mathbf{k}') = \mathbf{A}_1(\mathbf{k})\mathbf{B}_1(k'_1, k_2)\mathbf{A}_2(k'_1, k_2)\mathbf{B}_2(\mathbf{k}'). \quad (19)$$

If we change the multipliers in (18) then we also obtain a compact operator. ■

**Proof of Theorem 1.4.** Suppose that  $\mathbf{A}_0$  is non-invertible for some  $\mathbf{k}^0 \in [0, 1]^2$  with a corresponding null-vector  $\mathbf{f} \in \mathbb{C}^M$  having the unit Euclidean norm. Consider some sequence of characteristic functions  $\chi_n(\mathbf{k})$  of the sets  $\Omega_n \subset [0, 1]^2$ ,  $n \in \mathbb{N}$ , where domains  $\Omega_n$  tends to the point  $\mathbf{k}^0$ . Taking constants  $c_n = (\text{mes}\Omega_n)^{-\frac{1}{2}}$  we obtain that  $\mathbf{u}_n = c_n\chi_n\mathbf{f}$  have unit  $L_{2,M}^2$ -norm and

$$\mathbf{A}_0\mathbf{u}_n \rightarrow 0, \quad (20)$$

since  $\mathbf{f}$  is a null-vector of the continuous matrix-valued function  $\mathbf{A}_0$  at the point  $\mathbf{k}^0$ . At the same time, in [1] it is proved that  $\Omega_n$  can be chosen such that

$$\mathbf{A}_i\langle\mathbf{B}_i\mathbf{u}_n\rangle_i \rightarrow 0, \quad i = 1, 2. \quad (21)$$

Let  $\mathbf{u} \in L_{2,M}^2$  be some continuous function. Then the  $L_{2,M}^2$ -inner product (\* means Hermitian conjugation)

$$\langle\mathbf{u}^*\mathbf{u}_n\rangle_{12} = (\text{mes}\Omega_n)^{\frac{1}{2}}\mathbf{u}(\mathbf{k}^0)^*\mathbf{f} + o(1) \rightarrow 0 \quad (22)$$

since  $\Omega_n$  tends to the point  $\mathbf{k}^0$  and hence their Lebesgue measures tends to 0. Then  $\langle\mathbf{u}^*\mathbf{u}_n\rangle_{12} \rightarrow 0$  for any  $\mathbf{u} \in L_{2,M}^2$  since  $L_{2,M}^2$ -norm of  $\mathbf{u}_n$  is bounded (equal to 1). Since  $\mathcal{K}$  is compact we may assume that  $\mathcal{K}\mathbf{u}_n \rightarrow \mathbf{v}$ ,  $\mathbf{v} \in L_{2,M}^2$ . Then

$$0 = \lim\langle(\mathcal{K}^*\mathbf{v})^*\mathbf{u}_n\rangle_{12} = \lim\langle\mathbf{v}^*\mathcal{K}\mathbf{u}_n\rangle_{12} = \langle\mathbf{v}^*\mathbf{v}\rangle_{12} \quad (23)$$

or in other words

$$\mathcal{K}\mathbf{u}_n \rightarrow 0. \quad (24)$$

Identities (20), (14), and (24) leads to  $\mathcal{A}\mathbf{u}_n \rightarrow 0$  which with  $\|\mathbf{u}_n\| = 1$  ( $\|\cdot\|$  is  $L_{2,M}^2$  norm) means that  $\mathcal{A}$  is non-invertible (by Banach theorem about bounded inverse linear mappings).

Suppose that  $\mathbf{E}_0$  is invertible everywhere. Then  $\mathcal{A}$  and

$$\mathcal{A}_1 = (\mathbf{E}_0\cdot)^{-1} \circ \mathcal{A} = (\mathbf{C}_0\cdot) \circ \mathcal{A} = \mathcal{I} + \mathbf{C}_1\langle\mathbf{B}_1\cdot\rangle_1 + \mathbf{C}_2\langle\mathbf{B}_2\cdot\rangle_2 + (\mathbf{C}_0\cdot) \circ \mathcal{K} \quad (25)$$

are invertible or non-invertible simultaneously. If  $\mathbf{E}_1$  is non-invertible at some  $k_2^0 \in [0, 1]$  then as it is shown in [1] the operator  $\mathcal{I} + \mathbf{C}_1\langle\mathbf{B}_1\cdot\rangle_1$  is non-invertible and there exist domains  $\Omega_n \subset [0, 1]$  tending to  $k_2^0$  such that

$$(\mathcal{I} + \mathbf{C}_1\langle\mathbf{B}_1\cdot\rangle_1)\mathbf{u}_n \rightarrow 0, \quad (26)$$

where  $\mathbf{u}_n = c_n\chi_n\mathbf{C}_1\mathbf{f}$ ,  $\chi_n(\mathbf{k})$  is the characteristic function of the set  $[0, 1] \times \Omega_n$ ,  $\mathbf{f}$  is a null-vector of  $\mathbf{E}_1(k_2^0)$  with the unite Euclidean norm, and  $c_n$  are taken such that  $\|\mathbf{u}_n\| = 1$ . It is true that for some  $\tilde{k}_1$  the Euclidean norm of  $\mathbf{C}_1(\tilde{k}_1, k_2^0)\mathbf{f}$  is non-zero since otherwise

$$\mathbf{0} = \mathbf{E}_1(k_2^0)\mathbf{f} = \mathbf{f}. \quad (27)$$

Then we have

$$1 = \|\mathbf{u}_n\| \geq \delta(\text{mes}\Omega_n)^{\frac{1}{2}}c_n, \quad (28)$$

where the absolute constant  $\delta$  depends on the the Euclidean norm of  $\mathbf{C}_1(\tilde{k}_1, k_2^0)\mathbf{f}$  only (recall that  $\mathbf{C}_1$  is a continuous matrix-valued function). The following estimates are fulfilled

$$\|\mathbf{C}_2\langle\mathbf{B}_2\mathbf{u}_n\rangle_2\| \leq C(\text{mes}\Omega_n)c_n \leq (C/\delta)(\text{mes}\Omega_n)^{\frac{1}{2}}, \quad (29)$$

where  $C$  is an absolute constant depending on  $\mathbf{C}_1$ ,  $\mathbf{C}_2$ , and  $\mathbf{B}_2$ . The right-hand side of (29) tend to 0 since  $\Omega_n$  tend to the point  $k_2^0$ . Using this fact, (26), and  $(\mathbf{C}_0\cdot) \circ \mathcal{K}\mathbf{u}_n \rightarrow 0$  (which can be proved in the same manner as (24)) we deduce that the operator  $\mathcal{A}_1$  is non-invertible by Banach theorem about bounded inverse linear mappings.

Suppose that  $\mathbf{E}_1$  is invertible everywhere. Then it is not difficult to show that (see [1])

$$(\mathcal{I} + \mathbf{C}_1\langle\mathbf{B}_1\cdot\rangle_1)^{-1} = \mathcal{I} - \mathbf{C}_1\mathbf{E}_1^{-1}\langle\mathbf{B}_1\cdot\rangle_1. \quad (30)$$

Multiplying (25) by (30) we deduce that the operators  $\mathcal{A}_1$  and

$$\mathcal{A}_2 = \mathcal{I} + \mathbf{C}_2\langle\mathbf{B}_2\cdot\rangle_2 - \int_{[0,1]^2} \mathbf{D}_1(\mathbf{k}, \mathbf{k}') \cdot (\mathbf{k}') d\mathbf{k}' + (\mathcal{I} - \mathbf{C}_1\mathbf{E}_1^{-1}\langle\mathbf{B}_1\cdot\rangle_1) \circ (\mathbf{C}_0\cdot) \circ \mathcal{K} \quad (31)$$

are invertible or non-invertible simultaneously. Here  $\cdot$  means an operator argument (as usual). Now we can apply again the above arguments. If  $\mathbf{E}_2$  is non-invertible at some  $k_1^0 \in [0, 1]$  then the operator  $\mathcal{I} + \mathbf{C}_2\langle\mathbf{B}_2\cdot\rangle_2$  is non-invertible and hence  $\mathcal{A}_2$  is non-invertible. Suppose that  $\mathbf{E}_2$  is invertible everywhere. Then  $\mathcal{I} + \mathbf{C}_2\langle\mathbf{B}_2\cdot\rangle_2$  is invertible with

$$(\mathcal{I} + \mathbf{C}_2\langle\mathbf{B}_2\cdot\rangle_2)^{-1} = \mathcal{I} - \mathbf{C}_2\mathbf{E}_2^{-1}\langle\mathbf{B}_2\cdot\rangle_2. \quad (32)$$

Multiplying (31) by (32) we deduce that the operators  $\mathcal{A}_2$  and  $\mathcal{I} + \mathcal{K}_1$  are invertible or non-invertible simultaneously. ■

### 3. Example

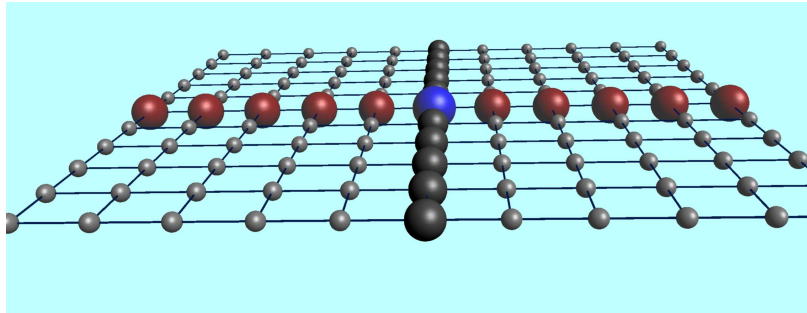


Figure 2: Simple 2D periodic lattice of springs and masses with two 1D defects and with one local defect.

Consider 2D spring-mass lattice with unite masses and unite Hook modules of springs. We add two perpendicular defects of masses  $1 + M_1 > 0$  and  $1 + M_2 > 0$ . The mass of cross point of guides is supposed to be  $1 + M_1 + M_2 > 0$ . Denoting anti-plane displacements at lattice points  $\mathbf{n} = (x, y) \in \mathbb{Z}^2$  as  $u_{\mathbf{n}}$ , we can write the equation of wave motion as

$$-(\Delta_{\text{discr}} u)_{\mathbf{n}} = \omega^2 u_{\mathbf{n}} + \omega^2 \begin{cases} M_1 u_{\mathbf{n}}, & \text{if } y = 0, x \neq 0, \\ M_2 u_{\mathbf{n}}, & \text{if } x = 0, y \neq 0, \\ (M_1 + M_2) u_{\mathbf{n}}, & \text{if } x = y = 0, \end{cases} \quad (33)$$

where  $\omega$  is a frequency, and the energy  $\omega^2$  plays the role of spectral parameter of our discrete periodic (Laplace  $\Delta_{\text{discr}}$ ) operator with defects. Applying Fourier-Floquet-Bloch transformation (here it is the Fourier series)

$$u(\mathbf{k}) = \sum_{\mathbf{n} \in \mathbb{Z}^2} e^{2\pi i \mathbf{n}^\top \mathbf{k}} u_{\mathbf{n}}, \quad \mathbf{k} = (k_1, k_2) \in [0, 1]^2 \quad (34)$$

we rewrite (33) as an integral operator

$$(4 - 2 \cos 2\pi k_1 - 2 \cos 2\pi k_2)u = \omega^2 u + \omega^2 M_2 \int_0^1 u dk_1 + \omega^2 M_1 \int_0^1 u dk_2 \quad (35)$$

or in our notations (1)

$$Au - \omega^2 M_2 \langle u \rangle_1 - \omega^2 M_1 \langle u \rangle_2 = 0 \quad (36)$$

with  $A = 4 - 2 \cos 2\pi k_1 - 2 \cos 2\pi k_2 - \omega^2$ . In fact, the problem consists of the determining the spectrum (extended eigenvalue problem for  $\omega^2$ ) of the operator from (36). The operator belongs to  $\mathcal{A}_{2,1}$  and we can use our results for this problem. Following Definition 1.3 introduce (we do not use bold fonts for scalars)

$$C_0 = A^{-1}, \quad E_0 = A, \quad C_1 = -\omega^2 M_2 A^{-1}, \quad C_2 = -\omega^2 M_1 A^{-1}, \quad (37)$$

$$E_1(k_2) = 1 - M_2 \langle A^{-1} \rangle_1 = 1 + \omega^2 M_2 \begin{cases} \frac{-1}{\sqrt{(2 \cos 2\pi k_2 - 4 + \omega^2)^2 - 4}}, & \text{if } \omega^2 < 2 - 2 \cos 2\pi k_2, \\ \frac{1}{\sqrt{(2 \cos 2\pi k_2 - 4 + \omega^2)^2 - 4}}, & \text{if } \omega^2 > 6 - 2 \cos 2\pi k_2. \end{cases} \quad (38)$$

If we take  $M_1, k_1$  instead of  $M_2, k_2$  in (38) then we obtain the identity for  $E_2(k_1)$ . Using the results of Theorem 1.4 we can describe the spectrum. The continuous part of the spectrum consists of three components. The first component  $\sigma_0$  (14) corresponds to the propagative waves without attenuation. So, the energy interval for such waves is

$$\sigma_0 = \{\omega^2 : A = 0 \text{ for some } \mathbf{k} \in [0, 1]^2\} = [0, 8]. \quad (39)$$

The second component  $\sigma_1$  (15) corresponds to the guided waves which propagate along the defect of masses  $1 + M_2$  and exponentially decay in perpendicular directions. The energy interval for such waves is

$$\sigma_1 = \{\omega^2 : E_1 = 0 \text{ for some } k_2 \in [0, 1]\} = \begin{cases} [\frac{4}{1-M_2^2}, \frac{6+2\sqrt{8M_2^2+1}}{1-M_2^2}], & M_2 < 0, \\ [0, \frac{-6+2\sqrt{8M_2^2+1}}{1-M_2^2}], & M_2 > 0. \end{cases} \quad (40)$$

The third component  $\sigma_2$  (15) has the same form as  $\sigma_1$  but with  $M_1$  instead of  $M_2$ . It is the energy interval for the guided waves that propagates along the defect of masses  $1 + M_1$ . All these continuous spectral components are the same as for the lattice with single line defects, see [8]. The new thing in our example is that the crossing of line defects can create the discrete spectrum. The discrete spectral component is

$$\sigma_3 = \{\omega^2 : \mathcal{I} + \mathcal{K}_1 \text{ is non-invertible}\}, \quad (41)$$

where (see (9),(10),(11), and (16))

$$\mathcal{K}_1 u = \int_{[0,1]^2} (D_2 - D_1)(\mathbf{k}, \mathbf{k}') u(\mathbf{k}') d\mathbf{k}', \quad u \in L_{2,1}^2 \text{ and} \quad (42)$$

$$D_1(\mathbf{k}, \mathbf{k}') = C_1(\mathbf{k}) E_1^{-1}(k_2) B_1(k'_1, k_2) C_2(k'_1, k_2) B_2(\mathbf{k}'), \quad (43)$$

$$D_2(\mathbf{k}, \mathbf{k}') = C_2(\mathbf{k}) E_2^{-1}(k_1) \int_0^1 B_2(k_1, k''_2) D_1(k_1, k''_2, \mathbf{k}') dk''_2. \quad (44)$$

Due to non-trivial kernels  $D_1, D_2$  the problem of presence or absence of eigenvalues can be complex and lengthy. Nevertheless, there are methods that allow to solve this problem effectively.

#### 4. Conclusion

In the current paper we extend some results from [1], [2] about the algebra of discrete periodic operators with parallel defects to the algebra of discrete periodic operators with perpendicular defects. We did this in the 2D case only. Even in 2D case we lost the explicit algorithm of finding discrete spectrum. Now it is not based on simple algebraic operations on some matrix-valued functions and few number of integrations as it was for parallel defects. The same thing is expected for multidimensional periodic operators with various crossing defects. While the situation is more or less clear in general (abstractly), the explicit algorithms of finding spectra corresponding to the crossing defects of lower dimensions are probably not exist (not so simple).

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